Stochastic calculus and control - tutorial

Stochastic Calculus

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Who am I? 😎

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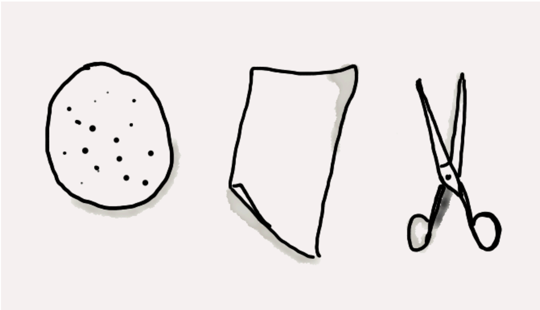
Why Randomness? 🤔



Bachelier model

First stochastic model for the stock price has been introduced in 1903 by `Louis Bachelier` in his PhD thesis under the supervision of the celebrated mathematician Henri Poincaré.

Louis Bachelier (1903)

● Bachelier is considered as the forefather of mathematical finance and a pioneer in the study of stochastic processes.

● PhD not well received because it attempted to apply mathematics to an unfamiliar area for mathematicians

● Although Bachelier's work on random walks predated Einstein's celebrated study of Brownian motion by five years, the pioneering nature of his work was recognized only after several decades, first by Andrey Kolmogorov

In the Bachelier model, the stock price is stochastic and follows a Brownian motion

$$S_t = S_0 + \sigma W_t, \quad t \leq T,$$

where $S_0 > 0$ is the initial price of the asset, T is the investement horizon, and $\sigma > 0$.

> What is a `Brownian motion` ?

1. Zooming out a random walk 🔍

Given a family $\{Y_i,\ i = 1, \ldots, n\}$ of n independent random variables with

$$\mathbb{P}[Y_i = 1] \ = \ 1 - \mathbb{P}[Y_i = -1] = \frac{1}{2}$$

we define the symmetric random walk

$$M_0 = 0 \text{ and } M_k \ = \sum_{j=1}^{k} Y_j \quad \text{for} \quad k = 0, \ldots, n\,.$$

By linear interpolation, define a continuous-time process:

$$M_t \ := \ M_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)\, M_{\lfloor t \rfloor + 1} \quad \text{for} \quad t \geq 0\,,$$

where $\lfloor t \rfloor$ denotes the largest integer less than or equal to t.

Define a stochastic process W^n by speeding up time and conveniently scaling:

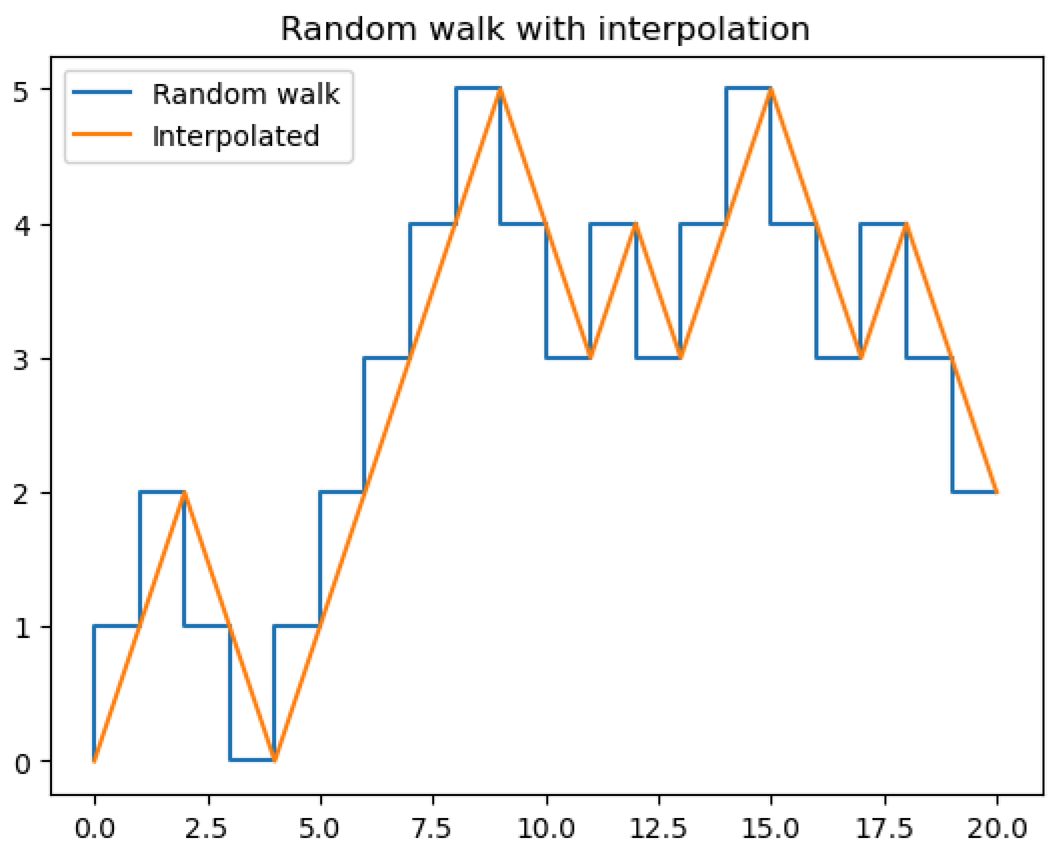
$$W^n_t \quad := \quad \frac{1}{\sqrt{n}} M_{nt}\,, \ \ t \geq 0\,.$$

1. Simulate trajectory of M and W^n for different n.

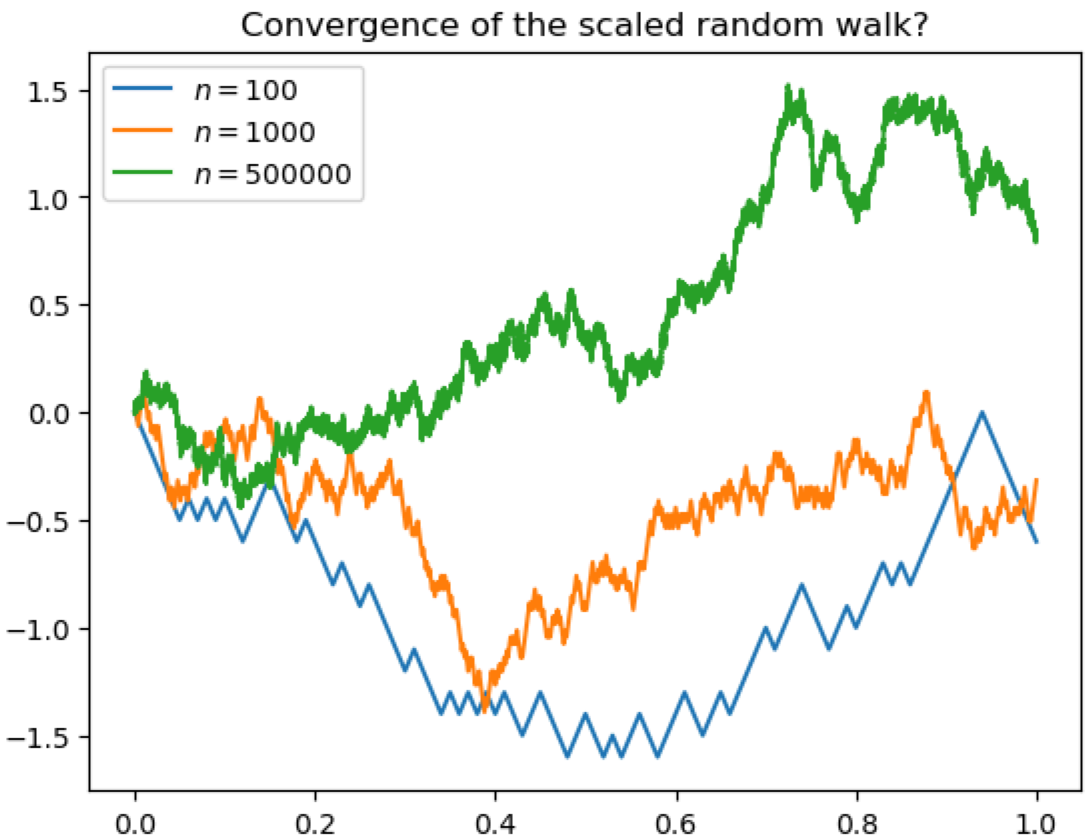
In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [25]:
```
#Simulate M and interpolate

# number of jumps
n = 20

#Fix random number generator seed
np.random.seed(1)
```

```
[0. 1. 2. 1. 0. 1. 2. 3. 4. 5. 4. 3. 4. 3. 4. 5. 4. 3. 4. 3. 2.]
```



In [31]:
```
#Simulate M and interpolate
```



We next set

$$t_k := \frac{k}{n} \quad \text{for} \quad k \in \mathbb{N}.$$

2. Prove the following properties of the process W^n:

- **Independence of increments** for $0 \leq i \leq j \leq k \leq \ell \leq n$, the increments $W^n_{t_\ell} - W^n_{t_k}$ and $W^n_{t_j} - W^n_{t_i}$ are independent,
- **First two moments** for $0 \leq i \leq k$, the two first moments of increment $W^n_{t_k} - W^n_{t_i}$ are given by

$$\mathbb{E}\left[W^n_{t_k} - W^n_{t_i}\right] = 0 \quad \text{and} \quad \mathbb{V}\text{ar}\left[W^n_{t_k} - W^n_{t_i}\right] = t_k - t_i.$$

  which shows in particular that the normalization by $n^{-1/2}$ in the definition of W^n prevents the variance from blowing up,
- **Martingality** with $\mathcal{F}^n_t := \sigma(Y_p, p \leq \lfloor nt \rfloor)$ for $t \geq 0$, the sequence $\left\{W^n_{t_k}, k \in \mathbb{N}\right\}$ is a discrete $\left\{\mathcal{F}^n_{t_k}, k \in \mathbb{N}\right\}$ −martingale:

$$\mathbb{E}\left(W^n_{t_k} \mid \mathcal{F}^n_{t_i}\right) = W^n_{t_i} \quad \text{for } 0 \leq i \leq k \leq n.$$

- **Quadratic variation**

$$\langle W^n, W^n \rangle_{t_k} := \sum_{j=1}^{k} \left(W^n_{t_j} - W^n_{t_{j-1}} \right)^2 = t_k.$$

3. Explain why at the limit $n \to \infty$ Gaussian distribution of the increments is expected to hold?

2. Brownian motion 🎲

Definition

Definition. (Brownian motion) Let $W = \{W_t,\ t \in \mathbb{R}_+\}$ be a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. W is a standard Brownian motion if

- $W_0 = 0$ and the sample paths $W_\cdot(\omega)$ are continuous for a.e. $\omega \in \Omega$,
- W has independent increments: for all $0 \leq t_1 < t_2 \leq t_3 < t_4$, $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent.
- For all $0 \leq s < t$, the distribution of $W_t - W_s$ is $N(0, t - s)$.

Some properties

The Brownian motion is the simplest example of a **Gaussian process**:

- A random variable $X \in \mathbb{R}$ is said to follow the **Gaussian distribution** $N(\mu, \sigma^2)$ if the law of X has a density given by

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- A random vector $(X_1, \ldots, X_n)$ is a **Gaussian vector** if for all $(a_1, \ldots, a_n) \in \mathbb{R}^n$, $\sum_{i=1}^n a_i X_i$ follows the Gaussian distribution.

- A stochastic process $(X_t)_{t \geq 0}$ is a **Gaussian process** if for all $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian vector.

- The law of a Gaussian process is completely characterized by its mean $m(t) = \mathbb{E}[X_t]$ and covariance $c(s, t) = \mathrm{Cov}[X_s, X_t]$.

Proposition. (Brownian motion is a Gaussian process) The Brownian motion is a centered Gaussian process with covariance function

$$c(s,t) = \mathbb{E}[W_s W_t] = t \wedge s.$$

Proposition. (Brownian motion is a Martingale) The Brownian motion is a Martingale with respect to its own filtration, i.e.

$$E[W_s|\mathcal{F}_t] = W_t, \quad s \geq t.$$

Exercise

Let B be a Brownian motion.

1. Prove that for all $t, s \geq 0$, $B_t \sim \mathcal{N}(0,t)$ and that $\mathbb{E}[B_s B_t] = \min(s,t)$

2. Let $\lambda \in \mathbb{R}$, prove that $(B_t)_{t\geq 0}$, $(B_t^2 - t)_{t\geq 0}$ and $(e^{\lambda B_t - \frac{\lambda^2}{2}t})_{t\geq 0}$ are martingales with respect to $\mathcal{F}_t := \sigma(B_t) := \sigma\{B_s : s \leq t\}$.

3. Let $t_0 = 0 < t_1 < \ldots < t_n = T$ be a subdivision of $[0,T]$ and $\delta = \max_{k\leq n-1} |t_{k+1} - t_k|$ (subdivision step).

   a. Let $X_k = (B_{k+1} - B_k)^2 - (t_{k+1} - t_k)$ for $k = 0, 1, \ldots, n-1$. Prove that

   $$\frac{\mathbb{E}[X_k^2]}{(t_{k+1} - t_k)^2}$$

   is independent of k.

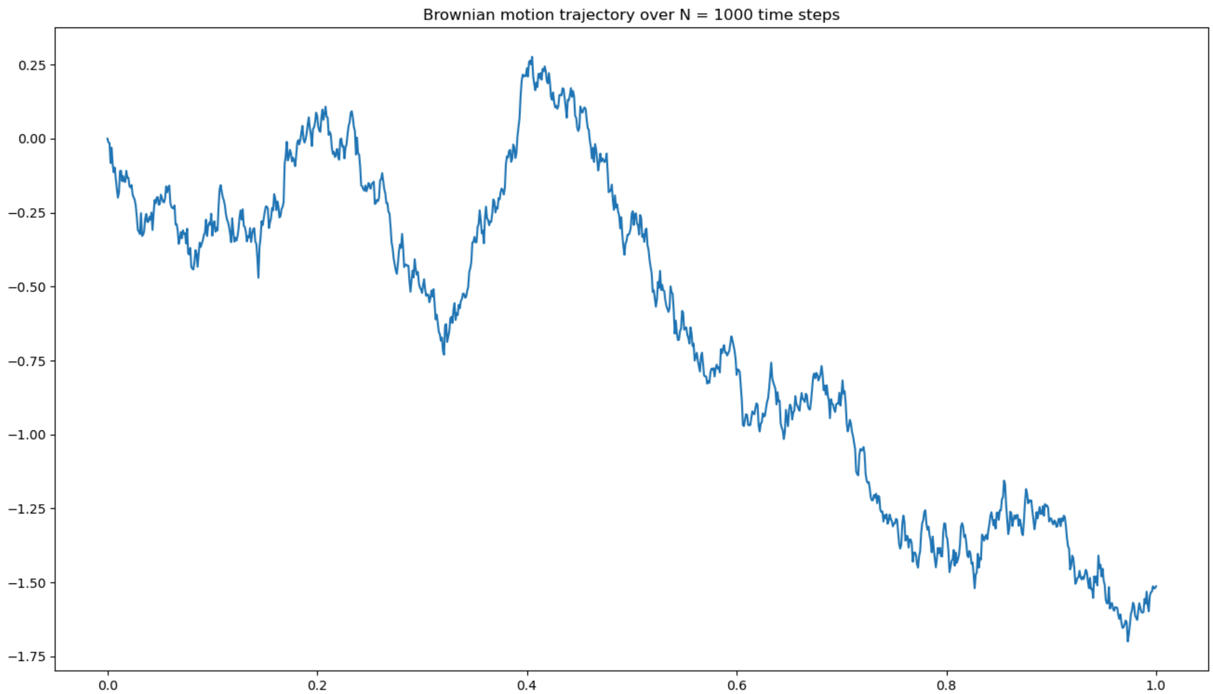
   b. Deduce that $\sum_{k=0}^{n-1} (B_{t_{k+1}} - B_{t_k})^2 \to T$ in L^2 when $\delta \to 0$.

3. Simulation 📈

1. Simulate a Brownian motion trajectory $(W_{t_0}, W_{t_1}, \ldots, W_{t_N})$ over the time gride $0 = t_0 < t_1 <\ldots t_{N-1} < t_N = T$, where

$$t_i = i\frac{T}{N}, \qquad i = 0, \dots, N.$$

In [28]:
```
np.random.seed(2)
T = 1.
N = 1000

time_grid = np.linspace(0, T, N+1)
```

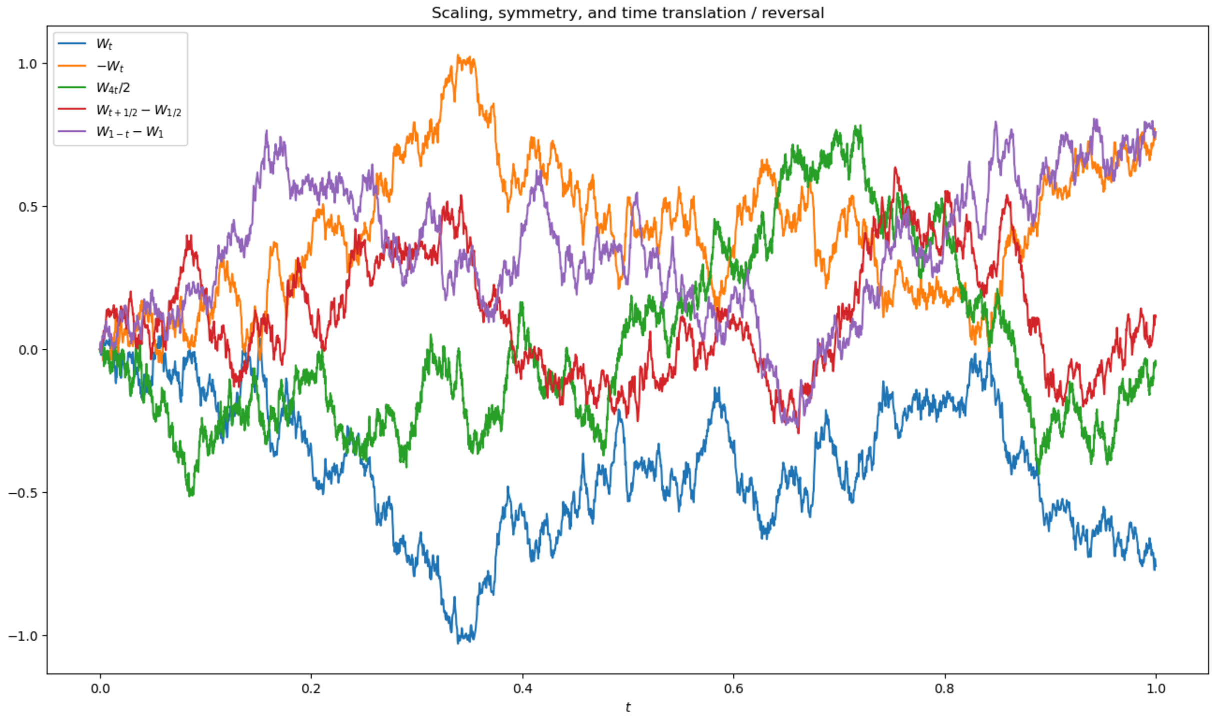


In [4]:
```
np.random.seed(2)
T = 1.
N = 10

time_grid = np.linspace(0, T, N+1)
```

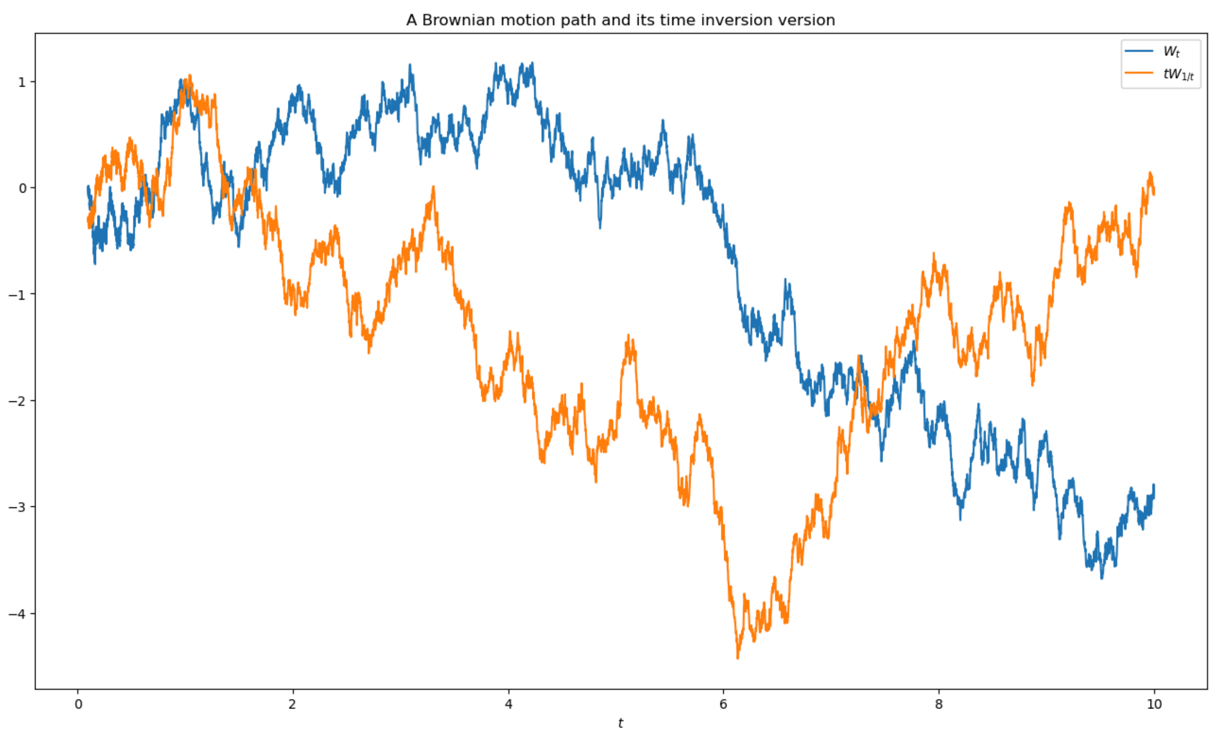
```
time_grid: [0.  0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1. ]
Z: [-0.41675785 -0.05626683 -2.1361961   1.64027081 -1.79343559 -0.84174737
  0.50288142 -1.24528809 -1.05795222 -0.90900761]
Y: [-0.1317904  -0.01779313 -0.67552452  0.51869917 -0.56713413 -0.26618389
  0.15902507 -0.39379467 -0.33455387 -0.28745345]
Y: [ 0.         -0.1317904  -0.01779313 -0.67552452  0.51869917 -0.56713413
 -0.26618389  0.15902507 -0.39379467 -0.33455387 -0.28745345]
```

2. Let $t_0 > 0$, and $c > 0$. Plot the following processes
   - $X^1 = (-W_t)_{t\geq 0}$
   - $X^2 = \left(c^{-1/2}W_{ct}\right)_{t\geq 0}$
   - $X^3 = (W_{t+t_0} - W_{t_0})_{t\geq 0}$
   - $X^4 = (W_{T-t} - W_T)_{t\geq 0}$
   - $X_0^5 := 0$ and $X_t^5 := tW_{1/t}, t > 0$

In [38]:
```
import numpy as np
import matplotlib.pyplot as plot
```



In [2]:



3. Based on the plots, what can you conjecture on these processes? Prove it.

4. Stochastic integral $\int$📐 dW

1. Show that the total (one) variation of the Brownian motion is infinite:

$$\lim_{n\to\infty}\sum_{i\geq 1}\left|W_{t_i^n\wedge t}-W_{t_{i-1}^n\wedge t}\right|=\infty\quad\text{in }L^2.$$

2. Let $(\theta)_{t\geq 0}$ be an elementary process.

   (i) Prove that $(\int_0^t \theta_s dW_s)_{t\geq 0}$ and $((\int_0^t \theta_s dW_s)^2-\int_0^t \theta_s^2 ds)$ are martingales with respect to the Brownian filtration.

   (ii) Compute $\mathbb{E}[(\int_0^T \theta_s dW_s)^2]$

   (iii) Assume that θ is determinstic. Provide the law of $\int_0^T \theta_s dW_s$.

3. Let f,g be a measurable function such that $\int_0^T f^2(s)ds<\infty$ and $\int_0^T g^2(s)ds<\infty$. What can you say about the distribution of $\int_0^T f(s)dW_s$? What about $\mathrm{Cov}(\int_0^T g(s)dW_s,\int_0^T f(s)dW_s)$?

5. Itô Vs Stratonovich ⚔️

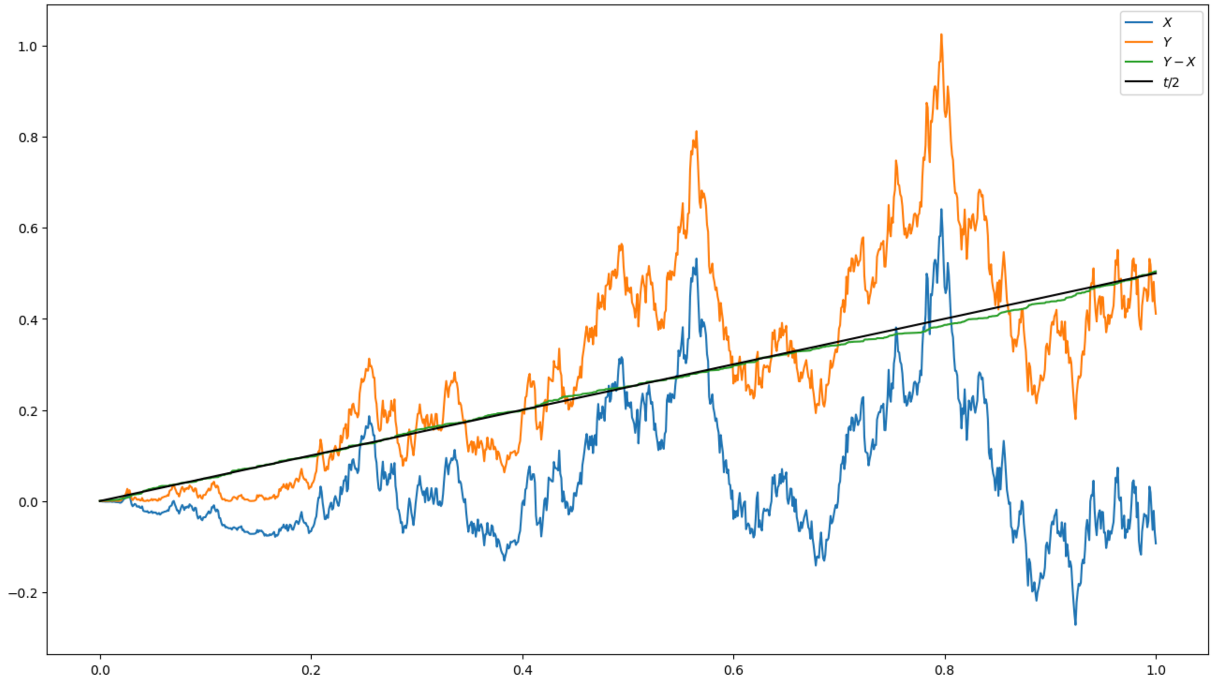
Let W be a Brownian motion and $t_i=i/T$, $i=0,1,\ldots,n$ a partition of $[0,T]$.

Denote by

$$X_t=\sum_{i=0}^{n-1}W_{t_i}(W_{t_{i+1}\wedge t}-W_{t_i\wedge t}),\quad Y_t=\sum_{i=0}^{n-1}\frac{W_{t_i}+W_{t_{i+1}}}{2}(W_{t_{i+1}\wedge t}-W_{t_i\wedge t}).$$

1. Plot on the same graph a sample path of X, Y, $(X-Y)$ and the function $t\mapsto t/2$.

In [20]:
```python
import numpy as np
import matplotlib.pyplot as plot
plot.rcParams['figure.figsize'] = [16, 9]
```



2. Comment. What can you conjecture on the relation between X, Y and $t \mapsto t$?
3. Prove the result mathematically.
4. By approximating W by $W_t^n = \sum_{i=0}^{n-1} W_{t_i} 1_{]t_i, t_{i+1}]}(t)$ compute $\int_0^T W_s dW_s$.

6. First look at Black-Scholes

No interest in the community in the work of Bachelier, although the Russian mathematician Kolmogorov has cited Bachelier... Growing interest starting 1960s, Robert Merton, Fisher Black and Myron Scholes model

Scholes received the 1997 Nobel Memorial Prize in Economic Sciences, Black wasn't awarded because he died in 1995.

Let W be a scalar Brownian motion and $S_t = S_0 e^{\int_0^t (r-\frac{\sigma^2}{2})(s)ds + \int_0^t \sigma(s)dW_s}$, for some constant $S_0 > 0$, and some deterministic functions r and σ.

1. Provide an explicit expression of $\mathbb{P}[S_T \geq K]$ for some $K > 0$ in terms of the cdf $\mathbf{N}$ of the standard $\mathcal{N}(0,1)$ distribution.
2. Provide an explicit expression of $\mathbb{E}[e^{-\int_0^T r(s)ds}(S_T - K)^+]$.
3. Prove that S solves a stochastic differential equation.

7. Itô's formula and heat equation 🔥

Use the Ito formula to write each of the following stochastic processes $(X_t)_{t\geq 0}$ as Itô processes:

1. $X_t = e^{t/2}\cos(W_t)$
2. $X_t = e^{t/2} sin(W_t)$
3. $X_t = (W_t + t)e^{-W_t - t/2}$

In [ ]:

8. Lévy's characterization of Brownian motion 🤯

Let W be a Brownian motion, and θ measurable with respect to $(\mathcal{F}_t^W)_{t\geq 0}$ be such that $\theta_t^2 = 1$ for all $t \geq 0$. Show that the process

$$X_t \ := \int_0^t \theta_s dW_s, \ \ j = 1, \ldots, n$$

is a Brownian motion with respect to the filtration $(\mathcal{F}_t^W)_{t\geq 0}$.

9. From the Ornstein-Uhlenbeck process to the square-root process

Fix a Brownian motion W and a and b reals. Consider the stochastic differential equation (SDE) :

$$dX_t = aX_t dt + bdW_t, \quad X_0 \in \mathbb{R}$$

1. Argue the existence and uniqueness of a solution.
2. Solve the SDE explicitely.
3. Simulate a trajectory of X for $a > 0$ and $a < 0$. What other methods can be used to simulate to trajectory of X? Comment.
4. What is the distribution of X_T?
5. Define $Y := X^2$. Derive the dynamics for Y
6. Show that Y is also a solution to an SDE.

In [ ]:

In [ ]:

References 📖

- Shreeve, S. E. (2004). Stochastic Calculus for Finance II: Continuous Time Models.